E 6378 (Pages : 3) Reg. No.....

Name.....

B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, MARCH 2019

Sixth Semester

Core Course—LINEAR ALGEBRA AND METRIC SPACES

(2013 Admission onwards)

Time : Three Hours Maximum Marks : 80

Part A (Objective Type Questions)

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Write the span $\{A_1, A_2\}$ where $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
2. What is the geometrical meaning "two vectors in $\mathbb{R}^2$ being linearly dependent" ?
3. Give the standard basis for the vector space P^3.
4. Define a linear transformation.
5. What is the nullity of a linear transformation $T : V \to W$ which is one to one ?
6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find TV for any V in $\mathbb{R}^2$.
7. Define the usual metric on the complex plane C.
8. Define a closed set in a metric space.
9. Define boundary point of a subset of a metric space.
10. Let X, Y be metric spaces. When is a function $f : X \to Y$ said to be continuous at x_0 in X.

(10 × 1 = 10)

Part B (Short Answer Questions)

Answer any **eight** *questions.*
Each question carries 2 *marks.*

11. Prove that additive inverse of a vector in a vector space is unique.
12. Show that $\{[a, b] : a = -b\}$ is a subspace of $\mathbb{R}^2$.

Turn over

13. Prove that a set of vectors in a vector space that contains the zero vector is linearly dependent.

14. Determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T[a, b] = [2a, 3b]$ is a linear transformation.

15. Find the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3b - a \end{bmatrix}$ with respect to the standard basis for $\mathbb{R}^2$.

16. Find the rank and nullity of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$.

17. Define a metric space and subspace of a metric space.

18. Prove that in a metric space the empty set and full set are open sets.

19. In a metric space prove that any intersection of closed sets is closed.

20. Prove that a subset A of a metric space is closed if and only if $A = \bar{A}$.

21. Define convergence of a sequence in a metric space.

22. Let X be a metric space with metric d defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ in $x \neq y$. Prove that any mapping of X into any other metric space is continuous.

$(8 \times 2 = 16)$

Part C (Short Essay Questions)

Answer any **six** *questions.*
Each question carries 4 *marks.*

23. Prove that a non-empty subset S of a vector space V is a subspace of V if and only if $\alpha u + \beta v \in S$ whenever $u, v \in S$ and α, β any two scalars.

24. Prove that $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$, considered as column vectors.

25. Find the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{P}^2$ defined by : $T\begin{bmatrix} a \\ b \end{bmatrix} = (4a + b)t^2 + (3a)t + (2a - b)$ with respect to the standard basis for $\mathbb{R}^2$ and the standard basis for $\mathbb{P}^2$.

26. Find a basis for the span of $S = \left\{ \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix} \right\}$.

27. Find the Kernel and image of the linear transformation $T : \mathbb{P}^2 \to M_{2\times 2}$ defined by :

$T\left(at^2+bt+c\right) = \begin{bmatrix} a & 2b \\ 0 & a \end{bmatrix}$. What is the dimension of $I_m(T)$?

28. Define the image of a linear transformation $T : V \to W$ and prove that it is a subspace of W.

29. In any metric space, prove that each closed sphere is a closed set.

30. Define a complete metric space and prove that every closed subspace of a complete metric space is complete.

31. Let X and Y be metric spaces and f a mapping of X into Y. If f is continuous prove that $f^{-1}(G)$ is open in X for every open set G of Y.

$(6 \times 4 = 24)$

Part D (Essay Questions)

Answer any **two** *questions.*
Each question carries 15 *marks.*

32. Prove that $\left\{ [x,y,z] \in \mathbb{R}^3 : x = 0 \right\}$ is a vector space under regular addition and scalar multiplication.

33. (a) Let T be a linear transformation from an n-dimensional vector space V into W and let $\{v_1, v_2, \ldots, v_k\}$ be a basis for the Kernel of T. If this basis is extended to a basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V, then prove that $\{Tv_{k+1}, T(v_{k+2}), \ldots, T(v_n)\}$ is a basis for $I_m(T)$.

(b) Deduce that rank (T) + nullity (T) = n, the dimension of V.

34. Prove that every non-empty open set on the real line is the union of a countable disjoint class of open intervals.

35. Let X be a metric space, let Y be a complete metric space, and let A be a dense subspace of X. If f is a uniformly continuous mapping of A into Y, prove that f can be extended uniquely to a uniformly continuous mapping g of X into Y.

$(2 \times 15 = 30)$